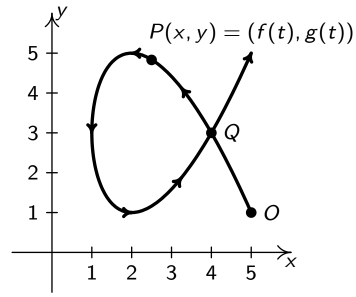
SECTION 12.1: PARAMETRIC EQUATIONS

In this section, we present a new concept which allows us to use functions to study these kinds of curves. To motivate the idea, we imagine a bug crawling across a table top starting at the point O and tracing out a curve C in the plane, as shown below.



The curve C does not represent y as a function of x because it fails the Vertical Line Test and it does not represent x as a function of y because it fails the Horizontal Line Test. However, since the bug can be in only one **place** $P(x, y)$ at any given time t, we can define the x-coordinate of P as a function of t and the y-coordinate of P as a (usually, but not necessarily) different function of t.

Traditionally, $f(t)$ is used for x and $g(t)$ is used for y. The independent variable t in this case is called a **parameter** and the system of equations

$$\begin{cases} x &= f(t) \\ y &= g(t) \end{cases}$$

is called a **system of parametric equations** or a **parametrization** of the curve C. (Note the use of the indefinite article 'a'. As we shall see, there are infinitely many different parametric representations for any given curve.) The parametrization of C endows it with an **orientation** and the arrows on C indicate motion in the direction of increasing values of t.

In this case, our bug starts at the point O, travels upwards to the left, then loops back around to cross its path. Here, the bug reaches the point Q at two different times. While this does not contradict our claim that $f(t)$ and $g(t)$ are **functions** of t, it shows that neither f nor g can be one-to-one. (Think about this before reading on.) at the point Q and finally heads off into the first quadrant.

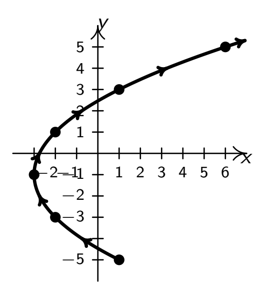
It is important to note that the curve itself is a set of points and as such is devoid of any orientation. The parametrization determines the orientation and as we shall see, different parametrizations can determine different orientations.

EXAMPLE 1: Sketch the curve described by $\begin{cases} x &= t^2 - 3 \\ y &= 2t - 1 \end{cases}$ for $t \geq -2$.

Ans: We follow the same procedure here as we have time and time again when asked to graph anything new – choose values for t, then plot and connect the corresponding points.

Since we are told $t \geq -2$, we start there and as we plot successive points, we draw an arrow to indicate the direction of the path for increasing values of t.

t	$x(t)$	$y(t)$	$(x(t), y(t))$
-2	1	-5	$(1, -5)$
-1	-2	-3	$(-2, -3)$
0	-3	-1	$(-3, -1)$
1	-2	1	$(-2, 1)$
2	1	3	$(1, 3)$
3	6	5	$(6, 5)$



The curve sketched out in above certainly looks like a parabola, and the presence of the t^2 term in the equation $x = t^2 - 3$ reinforces this hunch.

Since the parametric equations $\{x = t^2 - 3,\ y = 2t - 1$ given to describe this curve are a **system** of equations, we can use good old substitution to eliminate the parameter t and get an equation relating just x and y.

Solving the equation $y = 2t - 1$ for t, we get $t = \dfrac{y+1}{2}$. Substituting this into the equation $x = t^2 - 3$ yields:

$x = \left(\dfrac{y+1}{2}\right)^2 - 3$ or, after some rearrangement, $(y+1)^2 = 4(x+3)$.

Thinking back to College Algebra, we see that the graph of this equation is indeed a parabola with vertex $(-3, -1)$ which opens to the right, as required.

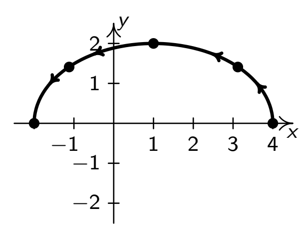
Technically speaking, the equation $(y+1)^2 = 4(x+3)$ describes the **entire** parabola, while the parametric equations $\{x = t^2 - 3,\ y = 2t - 1$ for $t \geq -2$ describe only a **portion** of the parabola.

Eliminating the parameter and obtaining an equation in terms of x and y, whenever possible, can be a great help in graphing curves determined by parametric equations. If the equations involved are algebraic, then using the substitution method described above can work wonders. If, on the other hand, the parametrization involves the trigonometric functions, the strategy changes slightly. In this case, it is often best to solve for the trigonometric functions and relate them using an identity.

EXAMPLE 2: Sketch the curve described by: $\begin{cases} x &= 1 + 3\cos(t) \\ y &= 2\sin(t) \end{cases}$ for $0 \leq t \leq \pi$

Ans: We set about graphing by plotting points.

t	$x(t)$	$y(t)$	$(x(t), y(t))$
0	4	0	$(4, 0)$
$\frac{\pi}{4}$	≈ 3.12	≈ 1.41	$\approx (3.12, 1.41)$
$\frac{\pi}{2}$	1	2	$(1, 2)$
$\frac{3\pi}{4}$	≈ -1.12	≈ 1.41	$\approx (-1.12, 1.41)$
π	-2	0	$(-2, 0)$



The curve reminds us of an ellipse, and to prove that, we need to eliminate the parameter. Since we have trigonometric functions involved, namely $\cos(t)$ and $\sin(t)$, we will use identities to do so.

Recall the Pythagorean Identity $\cos^2(t) + \sin^2(t) = 1$. Solving $x = 1 + 3\cos(t)$ for $\cos(t)$ we get $\cos(t) = \dfrac{x-1}{3}$.

Likewise, solving $y = 2\sin(t)$ for $\sin(t)$, we get $\sin(t) = \dfrac{y}{2}$.

Substituting these expressions into $\cos^2(t) + \sin^2(t) = 1$ gives

$$\left(\frac{x-1}{3}\right)^2 + \left(\frac{y}{2}\right)^2 = 1 \quad \text{or} \quad \frac{(x-1)^2}{9} + \frac{y^2}{4} = 1.$$

Hence, the curve traced out by the parametrization does indeed lie along an ellipse.

EXAMPLE 3: Find a parametrization for each of the following curves and check your answers.

1. $y = x^2$ from $x = -3$ to $x = 2$

   Ans: Since $y = x^2$ is written in the form $y = f(x)$, we let $x = t$ and $y = f(t) = t^2$. Since $x = t$, the bounds on t match precisely the bounds on x we get $\{x = t,\ y = t^2$ for $-3 \le t \le 2$.

   The check is almost trivial; with $x = t$ we have $y = t^2 = x^2$ as $t = x$ runs from -3 to 2.

2. The line segment which starts at $(2, -3)$ and ends at $(1, 5)$

   Ans: To parametrize line segment which starts at $(2, -3)$ and ends at $(1, 5)$, we make use of the formulas $x = x_0 + (\Delta x)t$ and $y = y_0 + (\Delta y)t$ for $0 \le t \le 1$. While these equations at first glance are quite a handful, they can be summarized as 'starting point + (displacement)t'.

   To find the equation for x, we have that the line segment **starts** at $x = 2$ and **ends** at $x = 1$. This means the **displacement** in the x-direction is $\Delta x = (1-2) = -1$. Hence, the equation for x is $x = 2+(-1)t = 2-t$.

   Similarly for y, we note that the line segment starts at $y = -3$ and ends at $y = 5$. Hence, the displacement in the y-direction is $\Delta y = (5 - (-3)) = 8$, so we get $y = -3 + 8t$.

   Putting together our answers for x and y, we get $\{x = 2 - t,\ y = -3 + 8t$ for $0 \le t \le 1$.

   To check, we can solve $x = 2 - t$ for t to get $t = 2 - x$. Substituting this into $y = -3 + 8t$ gives $y = -3 + 8t = -3 + 8(2 - x)$, or $y = -8x + 13$. We know this is the graph of a line, so all we need to check is that it starts and stops at the correct points.

   When $t = 0$, $x = 2-t = 2$, and when $t = 1$, $x = 2-t = 1$. Plugging in $x = 2$ gives $y = -8(2)+13 = -3$, for an initial point of $(2, -3)$. When $x = 1$, $y = -8(1) + 13 = 5$ for an ending point of $(1, 5)$, as required.

3. The left half of the ellipse $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$

   Ans: In the equation $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$, we use the Pythagorean Identity to get $x = 2\cos(t)$ and $y = 3\sin(t)$.

   The normal range on the parameter in this case is $0 \le t < 2\pi$, but since we only want the left half of the ellipse, we restrict t to angles which correspond to Quadrants II and III: $\frac{\pi}{2} \le t \le \frac{3\pi}{2}$.

   Hence, our final answer is $\{x = 2\cos(t),\ y = 3\sin(t)$ for $\dfrac{\pi}{2} \le t \le \dfrac{3\pi}{2}$.

   Substituting $x = 2\cos(t)$ and $y = 3\sin(t)$ into $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$ gives:

   $$\frac{4\cos^2(t)}{4} + \frac{9\sin^2(t)}{9} = 1 \iff \cos^2(t) + \sin^2(t) = 1,$$

   proving the points generated by the parametric equations $\{x = 2\cos(t),\ y = 3\sin(t)$ lie on $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$.

   Plotting points reveals that the restriction $\dfrac{\pi}{2} \le t \le \dfrac{3\pi}{2}$ generates the left half of the ellipse, as required.

4. The circle $x^2 + 2x + y^2 - 4y = 4$

Ans: After completing the squares, we get $(x+1)^2 + (y-2)^2 = 9$, or $\dfrac{(x+1)^2}{9} + \dfrac{(y-2)^2}{9} = 1$.

Using the identity $\cos^2(t) + \sin^2(t) = 1$, we identify $\cos(t) = \dfrac{x+1}{3}$ and $\sin(t) = \dfrac{y-2}{3}$.

Rearranging these last two equations, we get $x = -1 + 3\cos(t)$ and $y = 2 + 3\sin(t)$.

In order to complete one revolution around the circle, we let t range through the interval $[0, 2\pi)$, so our final answer is: $\{x = -1 + 3\cos(t),\ y = 2 + 3\sin(t)$ for $0 \leq t < 2\pi$.

To check our answer, we could eliminate the parameter by solving $x = -1 + 3\cos(t)$ for $\cos(t)$ and $y = 2 + 3\sin(t)$ for $\sin(t)$, invoking a Pythagorean Identity, and then manipulating the resulting equation in x and y into the original equation $x^2 + 2x + y^2 - 4y = 4$.

Instead, we opt for a more direct approach. We substitute $x = -1 + 3\cos(t)$ and $y = 2 + 3\sin(t)$ into the equation $x^2 + 2x + y^2 - 4y = 4$ and show that the latter is satisfied for all t such that $0 \leq t < 2\pi$.

$$
\begin{aligned}
x^2 + 2x + y^2 - 4y &= 4 \\
(-1 + 3\cos(t))^2 + 2(-1 + 3\cos(t)) + (2 + 3\sin(t))^2 - 4(2 + 3\sin(t)) &\stackrel{?}{=} 4 \\
1 - 6\cos(t) + 9\cos^2(t) - 2 + 6\cos(t) + 4 + 12\sin(t) + 9\sin^2(t) - 8 - 12\sin(t) &\stackrel{?}{=} 4 \\
9\cos^2(t) + 9\sin^2(t) - 5 &\stackrel{?}{=} 4 \\
9\left(\cos^2(t) + \sin^2(t)\right) - 5 &\stackrel{?}{=} 4 \\
9\,(1) - 5 &\stackrel{?}{=} 4 \\
4 &\stackrel{\checkmark}{=} 4
\end{aligned}
$$

Now that we know the parametric equations give us points on the circle, we can go through the usual analysis to show that the entire circle is covered as t ranges through the interval $[0, 2\pi)$.

CALCULUS AND PARAMETRIC EQUATIONS:

Suppose a curve C is described by a set of parametric equations $\{x = f(t),\ y = g(t)$

- $\dfrac{dx}{dt}$ is the rate of change of x with respect to t:

  That is: $\dfrac{dx}{dt} > 0$ indicates movement to the right; $\dfrac{dx}{dt} < 0$ indicates movement to the left.

- $\dfrac{dy}{dt}$ is the rate of change of y with respect to t.

  That is: $\dfrac{dy}{dt} > 0$ indicates movement upwards; $\dfrac{dy}{dt} < 0$ indicates movement downwards.

- If $\dfrac{dx}{dt} \neq 0$, $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$

  – If $\dfrac{dy}{dt} = 0$ but $\dfrac{dx}{dt} \neq 0$, we have a horizontal tangent.

  – If $\dfrac{dy}{dt} \neq 0$ but $\dfrac{dx}{dt} = 0$, we have a vertical tangent.

  – If both $\dfrac{dy}{dt} = 0$ and $\dfrac{dx}{dt} = 0$, the curve is ‘not smooth’ and pretty much anything can happen.

- $\dfrac{d^2y}{dx^2} = D_x\left[\dfrac{dy}{dx}\right] = \dfrac{D_t\,[dy/dx]}{dx/dt}$

EXAMPLE 3: Suppose a particle travels along a **cycloid** C traced out by the set if parametric equations:

$$\{x = t - \sin(t),\ y = 1 - \cos(t) \quad \text{for} \quad 0 \leq t \leq 4\pi.$$

1. Find $\dfrac{dx}{dt}$ and use this to determine when the particle is moving to the right and to the left.

   Ans: $\dfrac{dx}{dt} = 1 - \cos(t)$. From a Sign Diagram, we find: $\dfrac{dx}{dt} > 0$ on $(0, 2\pi)$ and $(2\pi, 4\pi)$.

   Hence the particle is moving to the right at all points except when $t = 0$, 2π, and 4π.

2. Find $\dfrac{dy}{dt}$ and use this to determine when the particle is moving upwards and downwards.

   Ans: $\dfrac{dy}{dt} = \sin(t)$. From a Sign (Sine?) Diagram, we find:

   - $\dfrac{dy}{dt} > 0$ on $(0, \pi)$ and $(2\pi, 3\pi)$ so the particle is traveling upwards during these intervals of t.
   - $\dfrac{dy}{dt} < 0$ on $(\pi, 2\pi)$ and $(2\pi, 4\pi)$ so the particle is traveling downwards during these intervals of t.

3. Locate any vertical and horizontal tangents and find points where the parametrization is not smooth.

   Ans: When $t = \pi$ and $t = 3\pi$, $\dfrac{dy}{dt} = 0$ but $\dfrac{dx}{dt} \neq 0$ meaning we have horizontal tangents at these points.

   Subbing $t = \pi$ and $t = 3\pi$ into $x = t - \sin(t)$, and $y = 1 - \cos(t)$ produces the points: $(\pi, 2)$ and $(3\pi, 2)$.

   Both $\dfrac{dx}{dt} = 0$ and $\dfrac{dy}{dt} = 0$ when $t = 0$, 2π, and 4π, so the parametrization is not smooth at these values.

   Subbing $t = 0$, 2π, and 4π into the parametric equations produces the points: $(0, 0)$, $(2\pi, 0)$ and $(4\pi, 0)$.

4. Find and simplify an expression for $\dfrac{dy}{dx}$ and use this to determine where C increasing and decreasing.

   How does this result compare your results with numbers 1 and 2?

   Ans: $\dfrac{dy}{dx} = \dfrac{\sin(t)}{1 - \cos(t)}$. From a Sign Diagram we find:

   - $\dfrac{dy}{dx} > 0$ on $(0, \pi)$ and $(2\pi, 3\pi)$ so the curve C is increasing over these intervals of t.
   - $\dfrac{dy}{dx} < 0$ on $(\pi, 2\pi)$ and $(3\pi, 4\pi)$ so the curve C is decreasing over these intervals of t.

   NOTE: These results can be found by 'combining' the answers from parts 1 and 2!

5. Find and simplify an expression for $\dfrac{d^2y}{dx^2}$ and use this to analyze the concavity of C.

   Ans: $\dfrac{d^2y}{dx^2} = \dfrac{-1}{(1-\cos(t))^2} < 0$ except for when it is undefined at $t = 0$, 2π, and 4π.

   This means the curve C is concave down on the intervals of t: $(0, 2\pi)$ and $(2\pi, 4\pi)$.

6. Find the equation of the tangent line to the point on the cycloid corresponding to $t = \dfrac{3\pi}{2}$.

   Ans: The equation of the tangent line is: $y = m_{\tan}(x - x_0) + y_0$.

   When $t = \dfrac{3\pi}{2}$, $x = \dfrac{3\pi}{2} + 1$ and $y = 1$ and $\dfrac{dy}{dx} = -1$.

   Hence, the tangent line is: $y = (-1)\left(x - \left[\dfrac{3\pi}{2} + 1\right]\right) + 1$ which simplifies to $y = -x + \dfrac{3\pi}{2} + 2$.

   Graphing C along with this line near $\left(\dfrac{3\pi}{2} + 1, 1\right)$ confirms our answer.

INTEGRALS AND AREA:

If $f(x) \geq 0$ is continuous on $[a, b]$, then $\displaystyle\int_a^b f(x)\,dx$ gives the area between the graph of $y = f(x)$ and the x-axis.

If $x(t_a) = a$ and $x(t_b) = b$, then $dx = x'(t)\,dt$ and we can write the integral as follows:

$$\int_a^b f(x)\,dx = \int_a^b y\,dx = \int_{t_a}^{t_b} y(t)\,x'(t)\,dt$$

EXAMPLE 4:

1. Show the area under one arch of the cycloid of the previous example is 3π units^2.

   Ans: $x = t - \sin(t)$ so $x'(t) = 1 - \cos(t)$ and $y(t) = 1 - \cos(t)$, so:

   $$\text{Area} = \int_0^{2\pi} (1-\cos(t))\,(1-\cos(t))\,dt = \int_0^{\pi} \left(1 - 2\cos(t) + \cos^2(t)\right)\,dt = \ldots = 3\pi \checkmark$$

   .

2. Show the area enclosed by the ellipse: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is πab units^2.

   HINT: Consider the parametrization: $x = a\cos(t)$, $y = b\sin(t)$.

   Ans: $x(t) = a\cos(t)$ so $x'(t) = -a\sin(t)$ and $y = b\sin(t)$.

   One quarter of the ellipse is traced out as $0 \leq x \leq a$.

   Since $x = 0$ corresponds to $t = \frac{\pi}{2}$ and $x = a$ corresponds to $t = 0$, we get:

   $$\text{Area} = 4\int_{\frac{\pi}{2}}^{0} (b\sin(t))\,(-a\sin(t))\,dt = \ldots = ab\int_0^{\frac{\pi}{2}} \sin^2(t)\,dt = \ldots = 4ab\left(\frac{\pi}{4}\right) = \pi ab \checkmark$$

EXAMPLE 5: (VIDEO) In this example, we're working on your deltoids - not the muscles, but the curve, C:

$$\begin{cases} x(t) &= 2\cos(t) + \cos(2t) \\ y(t) &= 2\sin(t) - \sin(2t) \end{cases}, \quad 0 \leq t \leq 2\pi.$$

1. Find $\dfrac{dx}{dt}$ and $\dfrac{dy}{dt}$ and use them to find points (x, y) where:

   (a) C has vertical tangents

   (b) C has horizontal tangents

   (c) C is not smooth

   HINT: Recall: $\sin(2t) = 2\sin(t)\cos(t)$ and $\cos(2t) = 2\cos^2(t) - 1$ ...

2. (a) Show $\dfrac{dy}{dx}$ simplifies to $\dfrac{\cos(t) - 1}{\sin(t)}$.

   (b) Find the intervals of t over which the curve C is increasing and decreasing.

3. (a) Show $\dfrac{d^2y}{dx^2}$ simplifies to $\dfrac{1}{2\sin(t)\,(\cos(t)+1)\,(2\cos(t)+1)}$.

   (b) Find the intervals of t over which the curve C is concave up / concave down.

4. Graph C. Check your answer using a graphing utility.

5. It turns out the area enclosed by this deltoid is 2π square units. Can you verify this?

HOMEWORK: Section 12.1: 13 - 77 every other odd, 103, 105